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# Pathological quotient singularities which are not log canonical in positive characteristic

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## Abstract

In characteristic zero, quotient singularities are log terminal. Moreover, we can check whether a quotient variety is canonical or not by using only the cyclic subgroups of the relevant finite group if the group does not have pseudo-reflections. In positive characteristic, a quotient variety is not log terminal, in general. In this paper, we give an example whose singularity cannot be determined by looking at proper subgroups.

## 1 Introduction

Quotient singularities form one of the most basic classes of singularities. They behave well in characteristic zero. In characteristic zero, any quotient variety has log terminal singularities. Moreover, for a finite group  $G \subset \mathrm{GL}(d, \mathbb{C})$  without pseudo-reflection, we say that  $G$  is small, we can use Reid-Shepherd-Barron-Tai criterion [2, Theorem 3.21] to know the singularity of  $\mathbb{C}^d/G$ . Namely, the following two conditions are equivalent.

- $\mathbb{C}^d/G$  is canonical (resp. terminal).
- $\mathbb{C}^d/C$  is canonical (resp. terminal) for all cyclic subgroup  $C$  of  $G$ .

In positive characteristic, if the given finite group is tame, then the quotient variety is again log terminal and we can use the Reid-Shepherd-Barron-Tai criterion.

## 2 Main Theorem

If the group is wild, there exists a quotient variety which is not log terminal. We give an even more pathological example.

**Theorem** (Main Theorem). Let  $C_3$  be the cyclic group of order three and  $C_3^2$  be the product of two copies of it. Suppose that the group  $C_3^2$  is embedded in  $\mathrm{SL}(3, K)$  and this embedding makes  $C_3^2$  small, where  $K$  is algebraically closed field of characteristic three. Then the quotient variety  $\mathbb{A}_K^3/C_3^2$  is not log canonical.

The pathological point of this example is that:

$$\mathbb{A}_K^3/C$$

is canonical for all cyclic subgroup  $C \subset C_3^2$  because  $\#C = 3$  for any cyclic subgroup  $C$ , see [3]. But

$$\mathbb{A}_K^3/C_3^2$$

is not log canonical. This is in contrast to the fact that, in characteristic zero, we can use the Reid-Shepherd-Barron-Tai criterion.

We give the proof of the main theorem by explicit computation. The all small action of  $C_3^2$  are given in [1]. The actions

are parametrized by  $a \in K \setminus \mathbb{F}_3, b \in K$ . Then, we can compute the explicit form of the quotient varieties  $X$  for each action of  $C_3^2$ . We will find that the quotient varieties are classified in two types separating by whether  $b = 0$  or not about the parameter  $b$  of the action. Finally, we construct the proper birational morphism  $Y \rightarrow X$  with exceptional divisors which discrepancy is smaller than  $-1$ , which shows the quotient varieties are not log canonical. This construction given by a few times blow up along the singular loci in each case.

## 3 Application

As an application of the main result, we give a criterion when a quotient variety associated to a small wild finite group is log terminal in dimension three and characteristic three. According to the criterion, we can judge the singularity of a quotient variety by seeing the order of the acting group.

**Corollary.** Let  $G$  be a wild small finite group of  $\mathrm{GL}(3, K)$  where  $K$  is an algebraically closed field. We write  $\#G = 3^r n$  where  $r, n$  are positive integer and  $n$  is not divided by three.

- (i) If  $r = 1$  then  $\mathbb{A}_K^3/G$  is log terminal.
- (ii) If  $r \geq 2$  then  $\mathbb{A}_K^3/G$  is not log canonical.

## References

- [1] H.E.A.E.Campbell, R.J.Shank, D.L.Wehlau, *Ring of invariants for modular representations of elementary abelian  $p$ -groups*, Transformation Groups Vol.18,No.1 (2013) pp.1-22
- [2] J.Kollár, *Singularities of the minimal model program*, Cambridge University Press. (2013)
- [3] T.Yasuda, *The  $p$ -cyclic McKay correspondence via motivic integration*, Compositio Math.150(2014), 1125-1168.